Soln :-

suppose, the infinite line through A is a source which emits fluid at the rate of $2\pi m \rho$ units of mass per unit length per unit time.

This is the mass flux per unit time per unit length of the infinite cylinder whose trace in the plane of flow in the closed curve $\mathcal{C}$ and m is the strength of the line source.

Now, let us take $\mathcal{C}$ to be the circle centre A with radius r.

since the speed of flow $\vec{q}$ is same everywhere on $\mathcal{C}$. The mass flux is $2\pi r \rho q$ per unit length.

$$2\pi m \rho = 2\pi r \rho q$$

$$m = rq.$$

$$\Rightarrow q = m/r.$$

The velocity potential on such a circle is given by,

$$\vec{q} = -\nabla\phi$$

where, $\phi = \phi(r)$

$$m/r = -\partial\phi/\partial r$$

$$m\,\partial r/r = -\partial\phi$$

$$\partial\phi = -m\,\partial r/r$$

$$\phi = -m\log r.$$

If we consider the point P on the circle having polar co-ordinates $P(r,\theta)$ w.r. to 'A' as origin and if $\overline{OP} = z = re^{i\theta}$ then,

$$\log r = \log|z|.$$

$$\phi = -m\log r$$

$$\therefore \phi = -m\log|z|.$$

Hence we see that the complex velocity potential of the line source through A of uniform strength m is given by,

$$w = -m \log z \Rightarrow z = re^{i\theta} \Rightarrow \log z = \log (re^{i\theta})$$

$$= \log((r) + i\theta)$$

$$w = -m \{ \log (r + i\theta) \}$$

$$\log mn = \log m + \log n$$

$$\psi(r, \theta) = -m\theta .$$

$$\phi(r, \theta) = -m \log r .$$

The stream lines ψ = constant shows that in any plane of flow are the straight lines θ = constant.

The equi-potential φ = constant $\Rightarrow r$ = constant which are concentric circles centered on 'A'.

NOTE :

If the line source of strength m. where, situated at $z = z_1$ instead of at $z = 0$ then $r = |z - z_1|$ and the complex velocity potential becomes $-m \log r$

$$w = -m \log |z - z_1|$$

$$\boxed{w = -m \log (z - z_1)}$$

LINE DOUBLETS :-

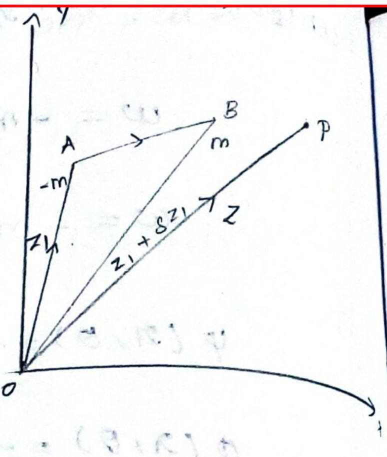


Figure shows a uniform line source of strength $-m$ at A, m at B and P be the fixed point all in the same plane of flow.

$\overline{OA} = z_1$, $\overline{OB} = z_1 + \delta z_1$, $\overline{OP} = z$

$\overline{AP} = \overline{OP} - \overline{OA}$

$\therefore \overline{AP} = z - z_1$

$\overline{BP} = \overline{OP} - \overline{OB}$

$\overline{BP} = z - z_1 - \delta z_1$

The complex potential at P due to the two line sources at A, B is,

$$w = -m \log (z - z_1)$$

$$w = -[-m \log (z - z_1) + m \log (z - z_1 - \delta z_1)]$$

$$w = m \log (z - z_1) - m \log (z - z_1 - \delta z_1) \quad \text{(1)}$$

using Taylor's expansion,

$$\log (z - z_1 - \delta z_1) = \log (z - z_1) - \delta z_1 \, \partial/\partial z [\log (z - z_1)] + O(\delta z_1)^2$$

$\log(\overline{z-z_1} - \delta z_1) = \log(z-z_1) - \delta z_1 \frac{1}{z-z_1} + O(\delta z)^2$ ②

Here we assume that $|\delta z_1|$ is very small

that higher powers $|\delta z_1|^2, |\delta z_1|^3, \ldots$ all

negligible.

∴ Approximating to the 1st order. we have,

using ② in ①,

$$w = m\log(z-z_1) - m\left\{\log(z-z_1) - \delta z_1 \frac{1}{z-z_1}\right\}$$

$$= m\log(z-z_1) - m\log(z-z_1) + m\,\delta z_1 (z-z_1)^{-1}$$

$$w = m\,\delta z_1 (z-z_1)^{-1} \rightarrow ③$$

If AB makes an angle α with OX then,

$$w = m|\delta z_1|\, e^{i\alpha}(z-z_1)^{-1}.$$

suppose that m becomes very large and

$|\delta z_1|$ is very small in such a way that,

$$\mu = m|\delta z_1|.$$

remains finite and constant.

Then the equal and opposite line sources at A are said to form a line doublet of strength μ per unit length $\overline{AB}$ gives the direction of the axis of the doublet

Hence the complex potential at P is given by,

$$w = \mu e^{i\alpha} (z - z_1)^{-1}.$$

PROBLEM:3 :-

Discuss the flow due to the uniform line doublet at 'o' of strength μ per unit length. It's axis being along ox.

solution :-

The complex velocity potential at $P(x, y)$ is, doublet $\Rightarrow w = \mu e^{i\alpha} (z - z_1)^{-1}$

$$w = \frac{\mu}{z} \qquad = \frac{\mu e^{i\alpha}}{(z - z_1)} \qquad \Rightarrow \alpha = 0,\ z_1 = 0$$

Axis lies along ox. $\quad = \frac{\mu \cdot e^{0}}{z} = \mu/z.$

$$w = \frac{\mu}{x+iy}$$

$$= \frac{\mu}{x+iy} \times \frac{x-iy}{x-iy}$$

$\Rightarrow (x+iy)(x-iy) = (x^2+y^2)$

$$= \mu \left[\frac{x-iy}{x^2+y^2} \right]$$

$$Re\, w = \phi(x,y) = \frac{\mu x}{x^2+y^2}$$

$$Im\, w = \phi(x,y) = -\frac{\mu y}{x^2+y^2}$$

The equipotentials are given by,

$\phi = \text{constant} \Rightarrow c = \frac{\mu x}{x^2+y^2} \Rightarrow x^2+y^2 = \mu x / c$

$x^2+y^2 = 2K_1 x$

(ie) $x^2+y^2 = 2K_1 x$.

which are co-axial circles having centre along the x-axis.

llly,

The stream lines ψ = constant are given by,

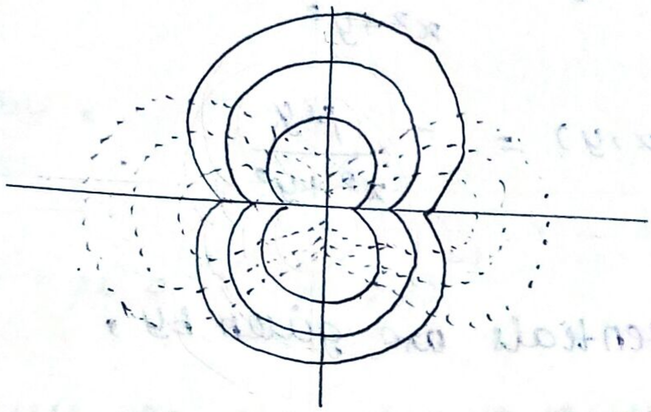
$x^2+y^2 = 2K_2 y \Rightarrow c = \frac{-\mu y}{x^2+y^2} \Rightarrow x^2+y^2 = \frac{-\mu}{c} y$

$x^2+y^2 = 2K_2 y$.

which are again co-axial circles.

The first ~~fabric~~ family has centre $(k_1, 0)$ radii k_1. The and second family has centre $(0, k_2)$ and radii k_2.

The two famillies are mutually orthogonal.



LINE VERTICES :-

For a two dimensional flow,

$$\vec{q} = u\vec{i} + v\vec{j}$$

where $u = u(x, y)$ and $v = v(x, y)$

then the vortocity vector is given by $\zeta = \nabla \times \vec{q}$.

$$\Rightarrow \zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k} \Rightarrow \hat{k} \Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

This shows that in two dimensional flow the vortocity vector is $\perp^{r}$ to the plane of flow.

PROBLEM : 4 :

Discuss the 2-dimensional flow for which the complex potential,

$$w = \left(\frac{iK}{2\pi}\right) \log z ,$$

where K is a real constant.

solution :

put $z = re^{i\theta}$

$$w = \frac{iK}{2\pi} \log (re^{i\theta})$$

$$= \frac{iK}{2\pi} [\log r + i\theta].$$

$$w = \frac{iK}{2\pi} \log r - \frac{K\theta}{2\pi}$$

equating the real and imaginary part.

we have,

$$\phi = \left(-\frac{K}{2\pi}\right)\theta.$$

$$\psi = \left(\frac{K}{2\pi}\right) \log r$$

ϕ = constant

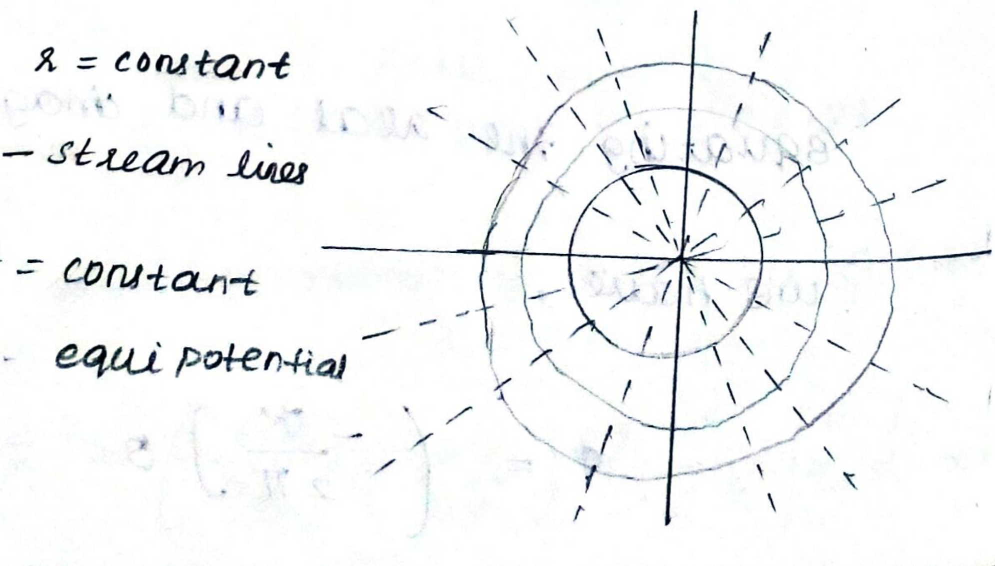
$\Rightarrow \theta$ = constant (or) the equi-potential are the radial vector to the origin.

IIIly,

ψ = constant

$\Rightarrow r$ = constant

(ie) The stream lines in the plane of flow of concentric circles r = constant and the two famillies are mutually orthogonal and both ϕ and ψ are Harmonic functions.



The radial and transverse components of the velocity are given by,

$$q_r = -\frac{\partial \phi}{\partial r} \quad ; \quad q_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$\Rightarrow q_r = 0 \quad ; \quad q_\theta = \left(\frac{K}{2\pi r}\right)$$

The circulation Γ around any closed curve C surrounding the origin and in the plane of flow is, given by,

$$\Gamma = \oint_C \vec{q} \cdot \vec{ds}$$

Taking $\vec{q}$ to the $\vec{q} = q_r \hat{r} + q_\theta \hat{\theta}$

$$= 0 + \left(\frac{K}{2\pi r}\right)\hat{\theta}$$

$$\vec{q} = \left(\frac{K}{2\pi r}\right)\hat{\theta}.$$

$$\vec{ds} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}.$$

$$\vec{q} \cdot \vec{ds} = \frac{Kr}{2\pi r}\,d\theta = \frac{K}{2\pi}\,d\theta.$$

$$\Gamma = \oint_{C} \frac{K}{2\pi}\, d\theta$$

$$= \frac{K}{2\pi} 2\pi$$

$$\Gamma = K$$

If C does not surround O (origin) then Γ is zero.

We have shown that a 2-dimensional distribution having a complex velocity potential,

$$\omega = \left(\frac{iK}{2\pi}\right) \log, \text{ where } K \text{ is a real constant}$$

gives a circulation Γ around any closed curve C in the plane of flow and enclosed in the origin 'o' and amount of K.

Also, any other curve in the plane of flow which does not enclosed 'o', the circulation is zero.

we now defined a uniform line vertex as follows,

"It is a uniform distribution along an infinite line such that the circulation around any curve C.

In any plane $\perp^r$ to that line is constant to K.

when C encloses the multiplication of the line and plane and its zero. when C does not contain the intersection".

The strength of uniform line vertex is defined to be K and its complex velocity potential is,

$\left(\frac{iK}{2\pi}\right) \log z$, where the origin is taken as multiplication of the plane with the line.

NOTE:

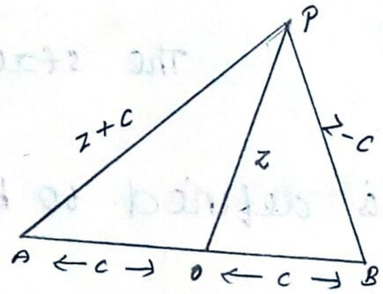
when the line intersects as $z = z_0$,

the complex potential is given by,

$$w = \left(\frac{iK}{2\pi}\right) \log (z - z_0).$$

PROBLEM : 5 :

Find the equations of the stream lines due to uniform line sources of strength m through the points, A $(-c, 0)$, B $(c, 0)$ and a uniform line sink of strength 2m through the origin.



solution :

Let P be the point such that

$$\overline{OP} = z = x + iy.$$

$$\therefore \overline{AP} = z + c$$

$$\overline{BP} = z - c$$

so the complex potential P due to the

sources A, B and the sink through the ~~region~~ origin is

given by,

$$w = -m \log (z - z_1)$$

$$w = -m \log (z+c) - m \log (z-c) + 2m \log z$$

$(a+b)(a-b) = a^2 - b^2$

$$= -m \log (z^2 - c^2) + m \log z^2$$

$$= m [\log z^2 - \log (z^2 - c^2)]$$

$$= m \log \frac{z^2}{z^2 - c^2} \quad [\because \log m - \log n = \log m/n]$$

$$= m \log \frac{(x+iy)^2}{(x+iy)^2 - c^2}$$

$$w = m \log \left(\frac{x^2 - y^2 + i\,2xy}{x^2 - y^2 - c^2 + i\,2xy} \right)$$

$$= m \log \left(\frac{x^2 - y^2 + i\,2xy}{x^2 - y^2 - c^2 + i\,2xy} \right) \times \left(\frac{x^2 y^2 - c^2 - i\,2xy}{x^2 - y^2 - c^2 - i\,2xy} \right)$$

$$= m \log \left(\frac{(x^2 - y^2)(x^2 - y^2 - c^2) - i(x^2 - y^2)\,2xy + i\,2xy(x^2 - y^2 - c^2) + 4x^2y^2)}{(x^2 y^2 - c^2)^2 + 4x^2y^2} \right)$$